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Sum of lengths of inversions in permutations

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Abstract

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1. Introduction

Let $\pi(1), \pi(2), \ldots, \pi(n)$ and $\sigma(1), \sigma(2), \ldots, \sigma(n)$ be two permutations of the set $\{1, 2, \ldots, n\}$. In connection with some order statistical investigations, Spearman (see [5], Ch. 2.) introduced the quantity

$$S(\pi, \sigma) = \sum_{i=1}^{n} (\pi^{-1}(i) - \sigma^{-1}(i))^2$$

and studied its properties. In the present paper we consider some similar quantities defined by permutations. One of them is the sum of lengths of inversions, that is,

$$\mathrm{SLI}(\pi, \sigma) = \sum_{\substack{1 \leqslant i < j \leqslant n \\ \pi^{-1}(i)\sigma(i) > \pi^{-1}(j)\sigma(j)}} (\pi^{-1}(i) \cdot \sigma(i) - \pi^{-1}(j)\sigma(j)).$$

On the other hand, we show some connections between $\mathrm{SLI}(\pi, \sigma)$ and a certain partial ordering of permutations.

2. Notations

The product sum of a permutation was introduced in [1]:

$$\mathrm{PS}(\pi) = \sum_{i=1}^{n} i \cdot \pi(i). \tag{1}$$

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Here we need an analogous concept for two permutations:

$$\mathrm{PS}(\pi, \sigma) = \sum_{i=1}^{n} \pi(i)\sigma(i). \tag{2}$$

Obviously, $\mathrm{PS}(\pi, \iota) = \mathrm{PS}(\pi)$, where ι is the identical permutation.

The complement π' of a permutation is defined by

$$\pi'(i) = n + 1 - \pi(i), \quad 1 \leqslant i \leqslant n. \tag{3}$$

A pair (i, j) $(1 \leqslant i < j \leqslant n)$ is called an inversion in π if $\pi(i) > \pi(j)$. The length of this inversion is $\pi(i) - \pi(j)$.

Accordingly, the sum of lengths of inversions in a permutation is:

$$\mathrm{SLI}(\pi) = \sum_{1 \leqslant i < j \leqslant n} d_{ij}(\pi), \tag{4}$$

where

$$d_{ij}(\pi) = \begin{cases} 0 & \text{if } \pi(i) < \pi(j) \\ \pi(i) - \pi(j) & \text{if } \pi(i) > \pi(j) \end{cases} \quad (i < j).$$

This can be formulated shortly as

$$\mathrm{SLI}(\pi) = \sum_{\substack{1 \leqslant i < j \leqslant n \\ \pi(i) > \pi(j)}} (\pi(i) - \pi(j)). \tag{5}$$

The more general concept for two permutations is:

$$\mathrm{SLI}(\pi, \sigma) = \mathrm{SLI}(\pi^{-1} \cdot \sigma). \tag{6}$$

It is obvious that

$$\mathrm{SLI}(\iota, \sigma) = \mathrm{SLI}(\iota^{-1}\sigma) = \mathrm{SLI}(\sigma).$$

Unfortunately, SLI is not a distance since it does not satisfy the triangle inequality for $n = 3$, $\pi = \iota$, $\sigma = (3, 1, 2)$, $\varrho = (2, 1, 3)$.

In the next section we prove some elementary statements.

Section 4 shows a connection between SLI and a certain partial ordering. The last section determines the expectation and the variance of $\mathrm{SLI}(\pi)$, where π is a random permutation.

3. Basic statements

Proposition 3.1.

$$\mathrm{PS}(\pi^{-1}) = \mathrm{PS}(\pi).$$

Proof. According to (1), we have

$$\mathrm{PS}(\pi) = 1 \cdot \pi(1) + 2 \cdot \pi(2) + \cdots + n \cdot \pi(n)$$

$$=\pi^{-1}(1)\cdot\pi(\pi^{-1}(1))+\pi^{-1}(2)\cdot\pi(\pi^{-1}(2))+\cdots+\pi^{-1}(n)\cdot\pi(\pi^{-1}(n))$$
$$=1\cdot\pi^{-1}(1)+2\cdot\pi^{-1}(2)+\cdots+n\pi^{-1}(n)=\mathrm{PS}(\pi^{-1}). \qquad \Box$$

Theorem 3.2.

$$\mathrm{SLI}(\pi)+\mathrm{PS}(\pi)=\frac{n(n+1)(2n+1)}{6}. \tag{7}$$

Proof. An elementary transposition $\tau_i=\tau_{i,i+1}$ is the permutation $(1,2,\ldots,i-1,i,i+1,i+2,i+3,\ldots,n)$. It is well known that any permutation π is a product of elementary transpositions. We prove the theorem by induction on the number of the elementary transpositions.

If this number is 0 then $\pi=\iota$. The assertion of the theorem is evidently true. Suppose that we proved it for any π being a product of k elementary transpositions ($k\geqslant 0$) and prove it for the product of $k+1$ elementary transpositions. It can obviously be written in the form $\pi\tau_i$, where π is the product of at most k elementary transpositions. $\pi\tau_i$ is the permutation obtained from π by exchanging the images $\pi(i)$ and $\pi(i+1)$ of i and $i+1$.

We have

$$\begin{aligned}\mathrm{PS}(\pi\tau_i)-\mathrm{PS}(\pi)&=i\cdot\pi(i+1)+(i+1)\cdot\pi(i)-i\pi(i)-(i+1)\pi(i+1)\\&=\pi(i)-\pi(i+1)\end{aligned} \tag{8}$$

since the other terms in $\mathrm{PS}(\pi\tau_i)$ and $\mathrm{PS}(\pi)$ are identical.

If $\pi(i)<\pi(i+1)$, τ_i creates a new inversion introducing the difference $\pi(i+1)-\pi(i)$.

If $\pi(i)>\pi(i+1)$, τ_i destroys the inversion between $\pi(i)$ and $\pi(i+1)$ whose corresponding difference was $\pi(i)-\pi(i+1)$.

In either case we have

$$\mathrm{SLI}(\pi\tau_i)-\mathrm{SLI}(\pi)=\pi(i+1)-\pi(i). \tag{9}$$

Now (8) and (9) imply

$$\mathrm{SLI}(\pi\tau_i)+\mathrm{PS}(\pi\tau_i)=\mathrm{SLI}(\pi)+\mathrm{PS}(\pi).$$

The latter one is equal to (7) by the induction hypothesis; so, it is also true for $\pi\tau_i$. $\Box$

Theorem 3.3.

$$\mathrm{SLI}(\pi,\sigma)+\mathrm{PS}(\pi,\sigma)=\frac{n(n+1)(2n+1)}{6}.$$

Proof. According to (2), Proposition 3.1 and (1), we have

$$\begin{aligned}\mathrm{PS}(\pi,\sigma)&=\sum_{i=1}^{n}\pi(i)\sigma(i)=\sum_{i=1}^{n}\pi(\pi^{-1}(i))\sigma(\pi^{-1}(i))\\&=\sum_{i=1}^{n}i\sigma(\pi^{-1}(i))=\mathrm{PS}(\pi^{-1}\cdot\sigma).\end{aligned}$$

Now applying Theorem 3.2 for $\pi^{-1}\sigma$ and (6), the assertion of the theorem is obtained. $\square$

4. Posets on permutations

The following poset is the so-called Bruhat weak order on permutations (see [2, 7, 10]).

Let π and σ be two permutations of the set $\{1, 2, \ldots, n\}$. We say that π covers σ and write

$$\pi \lessdot_{w} \sigma \text{ iff } \sigma = \pi\tau_i \text{ and } \pi(i) < \pi(i+1) \tag{10}$$

for some i $(1 \leqslant i < n)$.

Furthermore, let

$$\pi \leqslant_{w} \sigma \tag{11}$$

if

$$\pi = \pi_0 \lessdot \pi_1 \lessdot \cdots \lessdot \pi_m = \sigma \tag{12}$$

holds for some $m \geqslant 0$.

Figure 1 illustrates the Hasse diagram of the above poset for $n = 4$.

On the other hand, another ordering on the set of permutations of $\{1, 2, \ldots, n\}$ can be defined with the help of SLI:

$$\pi \leqslant \sigma \text{ iff } \mathrm{SLI}(\pi) \leqslant \mathrm{SLI}(\sigma). \tag{13}$$

The two orderings are connected by the following statement.

Proposition 4.1. SLI is a strictly increasing function on the poset defined by $\leqslant_{w}$.

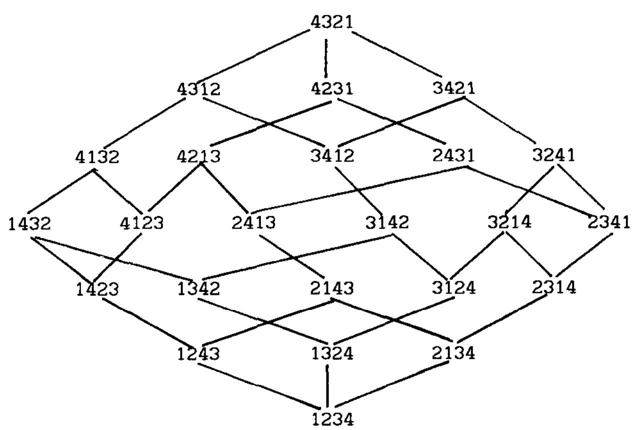

Fig. 1.

Proof. Suppose that $\pi \leqslant_{\mathrm{w}} \sigma$, that is, (12) holds with some $m \geqslant 0$. If $m=0$ then $\pi=\sigma$, $\mathrm{SLI}(\pi)=\mathrm{SLI}(\sigma)$ holds trivially. Prove $\mathrm{SLI}(\pi) \leqslant \mathrm{SLI}(\sigma)$ by induction on m.

Let τ_i be the last transposition:

$$\sigma=\pi_{m+1}=\pi_m \cdot \tau_i.$$

Then, by (9) and (10), we have

$$\mathrm{SLI}(\pi_{m+1})-\mathrm{SLI}(\pi_m)=\mathrm{SLI}(\pi_m \cdot \tau_i)-\mathrm{SLI}(\pi_m)=\pi_m(i+1)-\pi_m(i)>0.$$

This inequality, combined with the induction hypothesis $\mathrm{SLI}(\pi) \leqslant \mathrm{SLI}(\pi_m)$, proves the desired $\mathrm{SLI}(\pi) \leqslant \mathrm{SLI}(\pi_{m+1})$. It is easy to see that $<_{\mathrm{w}}$ implies strict inequality. $\square$

The above poset for $n=4$ is illustrated in Fig. 2.

5. Expectation and variance of the SLI of a random permutation

We need the following tools.

Lemma 5.1.

$$\mathrm{PS}(\pi)+\mathrm{PS}(\pi')=\frac{n(n+1)^2}{2}.$$

Proof. Let π and π' be two permutations of $\{1,2,\ldots,n\}$. According to (1) and (3), we have

$$\mathrm{PS}(\pi)+\mathrm{PS}(\pi')=\sum_{i=1}^{n} i \cdot \pi(i)+\sum_{i=1}^{n} i \cdot \pi'(i)=\sum_{i=1}^{n} i[\pi(i)+\pi'(i)]$$
$$=\frac{n(n+1)}{2} \cdot (n+1)=\frac{n(n+1)^2}{2}. \quad \square$$

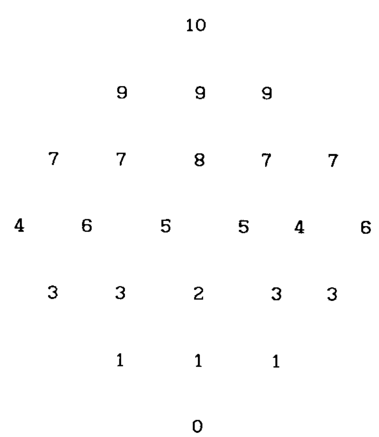

Fig. 2.

Proposition 5.2.

$$\mathrm{SLI}(\pi)+\mathrm{SLI}(\pi')=\binom{n+1}{3}.$$

Proof. We use Theorem 3.2 and Lemma 5.1:

$$\mathrm{SLI}(\pi)=\frac{n(n+1)(2n+1)}{6}-\mathrm{PS}(\pi),$$

$$\mathrm{SLI}(\pi')=\frac{n(n+1)(2n+1)}{6}-\mathrm{PS}(\pi')$$

and, hence,

$$\mathrm{SLI}(\pi)+\mathrm{SLI}(\pi')=\frac{n(n+1)(2n+1)}{3}-(\mathrm{PS}(\pi)+\mathrm{PS}(\pi'))$$

$$=\frac{n(n+1)(2n+1)}{3}-\frac{n(n+1)^2}{2}=\binom{n+1}{3}. \qquad \square$$

Theorem 5.3. *The expectation of the* SLI *of a random permutation is*

$$E(\mathrm{SLI}(\pi))=\frac{n(n+1)(n-1)}{12}=\frac{1}{2}\binom{n+1}{3}.$$

Proof.

$$E(\mathrm{PS}(\pi))=\frac{1}{n!}\sum_{\pi\in P_n}\mathrm{PS}(\pi)=\frac{1}{2n!}\left(\sum_{\pi\in P_n}(\mathrm{PS}(\pi)+\mathrm{PS}(\pi'))\right)$$

$$=\frac{1}{2n!}\cdot n!\,\frac{n(n+1)^2}{2}=\frac{n(n+1)^2}{4}, \tag{14}$$

where Lemma 5.1 is used. The statement of the theorem easily follows by (7). $\square$

Theorem 5.4.

$$D(\mathrm{SLI}(\pi))=\frac{n(n+1)\sqrt{n-1}}{12}.$$

Proof. It is obvious that $D(\mathrm{SLI}(\pi))=D(\mathrm{PS}(\pi))$ by (7).

To calculate $D(\mathrm{PS}(\pi))$, we need some elementary remarks.

If $i\neq k$ then the number of permutations π satisfying $\pi(i)=u$, $\pi(k)=v$ $(u\neq v)$ is $(n-2)!$. Therefore,

$$\sum_{\pi\in P_n}\pi(i)\pi(k)=(n-2)!\sum_{\substack{u=1\\ v=1\\ u\neq v}}^{n}u\cdot v=(n-2)!\left(\left(\sum_{u=1}^{n}u\right)^2-\sum_{u=1}^{n}u^2\right)$$

$$=(n-2)!\,\frac{n(n+1)}{2}\cdot\frac{(n-1)(3n+2)}{6}. \tag{15}$$

By similar considerations, one can see that

$$\sum_{\pi \in P_n} \pi^2(i) = (n-1)!\,\frac{n(n+1)(2n+1)}{6}. \tag{16}$$

Using (14) and (15), one can easily finish the calculation of

$$\begin{aligned}
E(\mathrm{PS}(\pi)^2) &= \frac{1}{n!}\sum_{i=1}^{n} i^2 \sum_{\pi \in P_n} \pi^2(i) + \frac{1}{n!}\sum_{\substack{1 \leqslant i \leqslant n\\ 1 \leqslant k \leqslant n\\ i \neq k}} ik \sum_{\pi \in P_n} \pi(i)\pi(k)\\
&= \frac{1}{n}\sum_{i=1}^{n} i^2 \cdot \frac{n(n+1)(2n+1)}{6}\\
&\quad + \frac{1}{n(n-1)}\sum_{\substack{1 \leqslant i \leqslant n\\ 1 \leqslant k \leqslant n\\ i \neq k}} ik\,\frac{n(n+1)}{2}\cdot\frac{(n-1)(3n+2)}{6}\\
&= \frac{1}{n}\left(\frac{n(n+1)(2n+1)}{6}\right)^2 + \frac{1}{n(n-1)}\left(\frac{n(n+1)}{2}\cdot\frac{(n-1)(3n+2)}{6}\right)^2\\
&= \frac{n^2(n+1)^2(9n^2+19n+8)}{144}.
\end{aligned} \tag{17}$$

Finally,

$$D^2(\mathrm{PS}(\pi)) = E((\mathrm{PS}(\pi))^2) - E^2(\mathrm{PS}(\pi))$$

can be determined by (17) and (14). □

6. Open problems

(1) Let P_n^k be the number of permutations of order n, having exactly k inversions. Then we know that

$$P_n^k = P_{n-1}^k + \cdots + P_{n-1}^{k-n+1}.$$

Give a closed formula for P_n^k. Our calculations show that

$$P_n^k = \binom{n+k-2}{k} - \binom{n+k-3}{k-2} + S_k,$$

where $S_1 = S_2 = S_3 = S_4 = 0$, $S_5 = \binom{n}{0}$, $S_6 = \binom{n}{1}$, $S_7 = \binom{n+1}{1} + \binom{n}{2}$ and $S_8 = 2\binom{n+1}{2} + \binom{n}{3}$ if $k \leqslant n$. For related formulas, see [6, 8].

(2) Determine the number of permutations π satisfying $\mathrm{PS}(\pi) = k$ for a given k. (It is known that this number is nonzero for an interval of integers.)

Lastly, we mention the following related works concerning metrics on permutations and some indices close to SLI: [3, 4, 9].

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